THE UNIVERSITY of EDINBURGH

Semidefinite Optimization Approaches for Reactive Optimal Power Flow Problems

Miguel F. Anjos

Chair of Operational Research, School of Mathematics, Univ. of Edinburgh, UK

Schöller Senior Fellow, University of Erlangen-Nürnberg, Germany

Various parts are joint work with **Christian Bingane** and Sébastien Le Digabel (Polytechnique Montréal), **Julie Sliwak** (RTE, LIPN-U. Sorbonne Paris Nord, and Polytechnique Montréal), Lucas Létocart and Emiliano Traversi (LIPN-U. Sorbonne Paris Nord), Manuel Ruiz (RTE), and Erling Andersen (MOSEK)

Institut de Recherche Technologique SystemX – Paris, France – 28 septembre 2022

Outline

1. Motivation, Background and our Contributions
2. Semidefinite Approach to AC Optimal Power Flow
3. Improved Clique Merging for Chordal Relaxations
4. Reactive Optimal Power Flow with Practical Constraints
5. Branch-and-Bound Algorithm & Computational Results
6. Summary and Research Challenges

AC optimal power flow (ACOPF)

- The optimal power flow (OPF) problem consists in finding a network operating point that optimizes a certain objective such as generation cost, active power losses, etc.
- Solving the OPF problem for general alternating current (AC) networks is difficult, in particular because of the nonconvex power flow constraints.
- ACOPF was first formulated in Carpentier (1962).
- Testing feasibility for an AC power flow system is strongly NP-hard (Bienstock and Verma, 2015).
- Local optima can occur if the feasible region is disconnected and/or because of the nonlinearities, and standard nonlinear solvers are shown to converge to these local optima (Bukhsh et al., 2013).
- Lack of fast and robust solution techniques.

ACOPF: A Major and Important Challenge

- Computational challenge: Find a global optimal solution with running time up to 3 to 5 orders of magnitude faster than existing solvers.
- Such a solution could potentially save $10G annually (Cain et al., 2012).
- ARPA-E Grid Optimization Competition: `https://gocompetition.energy.gov/`
  - Challenge 1: Security-constrained ACOPF (SCOPF) – Results announced in February 2020.
  - Challenge 2: SCOPF plus price-responsive demand, ramp rate constrained generators and loads, fast-start unit commitment, adjustable transformer tap ratios, phase shifting transformers, and switchable shunts.

    $\Rightarrow$ Reactive Optimal Power Flow
  - Challenge 2 – Monarch of the Mountain: January to October 2022
  - Challenge 3 – Starting

Reactive Optimal Power Flow

- The Reactive Optimal Power Flow (ROPF) problem, or Optimal Reactive Power Dispacth, is an ACOPF problem with discrete control devices for regulating reactive power.
- ROPF usually focuses on two means of action: activation of shunts and adjustment of transformer ratios.
- Because these additional requirements require the use of binary and/or integer variables, ROPF is generally more difficult than ACOPF.

Our Contributions

1. We propose an improved clique merging algorithm for semidefinite chordal relaxations of ACOPF.
2. We propose four semidefinite relaxations for ROPF, including a new tight convex model for tap changers, and solve instances of ROPF with more than 3000 buses.
3. We then consider a simpler version of ROPF and introduce some practical aspects.
4. We propose a flexible semidefinite optimization-based B&B method that is able to reach global optimality. To the best of our knowledge, this is the first time that a B&B approach is tested for ROPF.
5. We show that our method closes the gap or at least improve it on most instances, including large instances with 6000 to 10000 buses.
6. We show that the proposed methods find feasible solutions that are better than those obtained by rounding heuristics from the literature.

ACOPF Formulation

Minimize generation cost: $\sum_{k\in\mathcal{N}} c_{Gk} p_{Gk}$ subject to

1. Power balance equations:

$$\sum_{g\in\mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k |\mathrm{v}_k|^2 = \sum_{\ell=(k,m)\in\mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k)\in\mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{N},$$

$$\sum_{g\in\mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k |\mathrm{v}_k|^2 = \sum_{\ell=(k,m)\in\mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k)\in\mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{N},$$

2. Line flow equations:

$$\frac{\mathrm{v}_k}{\mathrm{t}_\ell}\left[\left(\mathrm{j}\frac{b'_\ell}{2} + \mathrm{y}_\ell\right)\frac{\mathrm{v}_k}{\mathrm{t}_\ell} - \mathrm{y}_\ell \mathrm{v}_m\right]^* = p_{f\ell} + \mathrm{j}q_{f\ell} \quad \forall \ell = (k,m) \in \mathcal{L},$$

$$\mathrm{v}_m\left[-\mathrm{y}_\ell\frac{\mathrm{v}_k}{\mathrm{t}_\ell} + \left(\mathrm{j}\frac{b'_\ell}{2} + \mathrm{y}_\ell\right)\mathrm{v}_m\right]^* = p_{t\ell} + \mathrm{j}q_{t\ell} \quad \forall \ell = (k,m) \in \mathcal{L},$$

3. Generator power capacities: $\underline{p}_{Gg} \leq p_{Gg} \leq \overline{p}_{Gg}$, $\underline{q}_{Gg} \leq q_{Gg} \leq \overline{q}_{Gg} \quad \forall g \in \mathcal{G}$,
4. Line thermal limits: $|p_{f\ell} + \mathrm{j}q_{f\ell}| \leq \overline{s}_\ell$, $|p_{t\ell} + \mathrm{j}q_{t\ell}| \leq \overline{s}_\ell \quad \forall \ell \in \mathcal{L}$,
5. Voltage magnitude limits: $\underline{v}_k \leq |\mathrm{v}_k| \leq \overline{v}_k \quad \forall k \in \mathcal{N}$,
6. Phase angle difference limits: $|\angle \mathrm{v}_k - \angle \mathrm{v}_m| \leq \overline{\delta}_\ell \quad \forall \ell = (k,m) \in \mathcal{L}$,
7. Reference bus constraint: $\angle \mathrm{v}_1 = 0$.

Solving ACOPF

There are (at least) three ways to tackle ACOPF:

1. Employ a nonlinear solver to find a local optimum.
2. Use a linear approximation of the power flow equations.
3. **Exploit convex relaxations of nonconvex constraints**.

Convex Approaches to ACOPF

- Various linear optimization relaxations, including the basic DC approximation, and a specifically designed quadratic convex relaxation.
- Spatial B&B algorithms based on convex relaxations, such as: QO (Godard et al., 2019), SOCO (Kocuk et al., 2017), SDO (Gopalakrishnan et al., 2012).
- Lasserre hierarchy of relaxations (Josz et al., 2015).
- Dozens of papers on semidefinite and related approaches to ACOPF, see the recent survey
  - Zohrizadeh, Josz, Jin, Madani, Lavaei, Sojoudi (2020). A survey on conic relaxations of optimal power flow problem. EJOR 287(2), 391-409.

Matrix Formulation of ACOPF

- ACOPF can be cast as a nonconvex polynomial optimization problem in complex variables.
- The nonlinearities in the ACOPF formulation only involve the voltage variables v_k.
- Use the vector of voltages $\mathbf{v}$ to define the matrix variable V in complex numbers:

$$\mathrm{V} = \mathbf{v}\mathbf{v}^H \Rightarrow \begin{cases} \mathrm{V}_{kk} = |\mathrm{v}_k|^2, & k \in \mathcal{N}, \\ \mathrm{V}_{km} = \mathrm{v}_k \mathrm{v}_m^*, & (k,m) \in \mathcal{L}. \end{cases}$$

- We can now use the individual entries of the variable V to linearize the quadratic terms in the ACOPF.

- We have developed a Julia module allowing the representation of polynomial problems in complex formulation (Sliwak et al., 2019): `https://ieeexplore.ieee.org/abstract/document/8810960`

Matrix Formulation of ACOPF (ctd)

- Power balance equations:

$$\sum_{g\in\mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k \mathrm{V}_{kk} = \sum_{\ell=(k,m)\in\mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k)\in\mathcal{L}} p_{t\ell} \quad \forall k \in \mathcal{N},$$

$$\sum_{g\in\mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k \mathrm{V}_{kk} = \sum_{\ell=(k,m)\in\mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k)\in\mathcal{L}} q_{t\ell} \quad \forall k \in \mathcal{N},$$

- Line flow equations:

$$\frac{1}{|\mathrm{t}_\ell|^2}\left(-\mathrm{j}\frac{b'_\ell}{2} + \mathrm{y}^*_\ell\right)\mathrm{V}_{kk} - \frac{\mathrm{y}^*_\ell}{\mathrm{t}_\ell}\mathrm{V}_{km} = p_{f\ell} + \mathrm{j}q_{f\ell} \quad \forall \ell = (k,m) \in \mathcal{L},$$

$$-\frac{\mathrm{y}^*_\ell}{\mathrm{t}^*_\ell}\mathrm{V}^*_{km} + \left(-\mathrm{j}\frac{b'_\ell}{2} + \mathrm{y}^*_\ell\right)\mathrm{V}_{mm} = p_{t\ell} + \mathrm{j}q_{t\ell} \quad \forall \ell = (k,m) \in \mathcal{L},$$

- Voltage magnitude limits: $\underline{v}_k{}^2 \le \mathrm{V}_{kk} \le \overline{v}_k{}^2 \quad \forall k \in \mathcal{N},$
- Phase angle diff. limits: $|\,\mathrm{Im}(\mathrm{V}_{km})| \le \mathrm{Re}(\mathrm{V}_{km}) \tan \overline{\delta}_\ell \quad \forall \ell = (k,m) \in \mathcal{L},$
- Matrix variable definition: $\mathrm{V} = \mathbf{v}\mathbf{v}^H$

Basic Semidefinite Relaxation of ACOPF

From the matrix formulation, we obtain the basic semidefinite relaxation (SDR) of ACOPF by relaxing the matrix constraint (Bai et al., 2008):

$$\begin{aligned} \mathrm{V} = \mathbf{v}\mathbf{v}^H &\Leftrightarrow \mathrm{V} \succeq 0 \text{ and } \operatorname{rank} \mathrm{V} = 1 \\ &\Rightarrow \mathrm{V} \succeq 0. \end{aligned}$$

- If the optimal solution of the SDR relaxation is a rank-one matrix, we have zero optimality gap and we can recover the globally optimal voltage profile.
- Lavaei and Low (2012) observed (in a different way) that the global optimal solution could be recovered using the SDR for several standard IEEE benchmarks.
- Lesieutre et al. (2011) showed that this failed for some practical cases but confirmed that the semidefinite approach was promising for identifying large numbers of solutions to the power flow equations.
- Main limitation of SDR: Computationally very expensive for large-scale networks.

Chordal Relaxations

- A matrix completion approach to exploit sparsity in semidefinite optimization was proposed by Fukuda et al. (2001).

Idea: Replace the PSD constraint on the large matrix V by PSD constraints on r smaller matrices:

$$\mathrm{V} \succeq 0 \Leftrightarrow X_i \succeq 0, i = 1..r \text{ with } n_i \ll n$$

plus constraints linking the entries common to two or more matrices X_i.

- Equivalence holds if and only if the graph representing the sparsity pattern of V is chordal.
- This motivates the idea of chordal relaxations for ACOPF, first proposed by Jabr (2012).
- Molzahn et al. (2013) first proposed clique merging for ACOPF.
- An improved clique merging for ACOPF was proposed by Sliwak et al. (2021) using a different criterion to estimate the computational cost.

Reactive Optimal Power Flow

Because of the presence of discrete variables, Reactive Optimal Power Flow (ROPF) is generally more difficult than the ACOPF.

We present here the following recent developments:

- Bingane et al. (2019) proposed 4 different semidefinite relaxations for ROPF, as well as a new tight convex model for tap changers.
- Sliwak et al. (2021) developed a full B&B approach for the problem with shunts, and with additional practical considerations.

ROPF Formulation

The formulation is the same as for ACOPF with the addition of binary variables for power balance at the buses $k \in \mathcal{U}$ where shunts are present:

$$\sum_{g \in \mathcal{G}_k} p_{Gg} - p_{Dk} - g'_k u_k \, |\mathrm{v}_k|^2 = \sum_{\ell=(k,m)\in\mathcal{L}} p_{f\ell} + \sum_{\ell=(m,k)\in\mathcal{L}} p_{t\ell}$$

$$\sum_{g \in \mathcal{G}_k} q_{Gg} - q_{Dk} + b'_k u_k \, |\mathrm{v}_k|^2 = \sum_{\ell=(k,m)\in\mathcal{L}} q_{f\ell} + \sum_{\ell=(m,k)\in\mathcal{L}} q_{t\ell}$$

with shunt variables $u_k \in \{0,1\} \; \forall k \in \mathcal{U}$, and changing the parameters t_ℓ to variables on the line flow equations for the lines $\ell = (k,m) \in \mathcal{T}$ where tap changers are present:

$$\frac{\mathrm{v}_k}{\mathrm{t}_\ell}\left[\left(\mathrm{j}\frac{b'_\ell}{2} + \mathrm{y}_\ell\right)\frac{\mathrm{v}_k}{\mathrm{t}_\ell} - \mathrm{y}_\ell \mathrm{v}_m\right]^* = p_{f\ell} + \mathrm{j}q_{f\ell}$$

$$\mathrm{v}_m\left[-\mathrm{y}_\ell\frac{\mathrm{v}_k}{\mathrm{t}_\ell} + \left(\mathrm{j}\frac{b'_\ell}{2} + \mathrm{y}_\ell\right)\mathrm{v}_m\right]^* = p_{t\ell} + \mathrm{j}q_{t\ell}$$

with tap ratios $\mathrm{t}_\ell \in \{\underline{\mathrm{t}}_\ell, \ldots, \bar{\mathrm{t}}_\ell\} \; \forall \ell \in \mathcal{T}$.

Semidefinite Relaxation 1 (SDR1)

Let

$$
\begin{aligned}
\mathrm{V} &:= \mathbf{v}\mathbf{v}^H, \\
\mathrm{w}_\ell &:= \frac{\mathrm{v}_k}{t_\ell} && \forall \ell = (k,m) \in \mathcal{T}. \\
\mathrm{W}_{\{\ell\}} &:= \begin{bmatrix} \mathrm{V}_{kk} & \mathrm{W}_{k\ell} & \mathrm{V}_{km} \\ \mathrm{W}^*_{k\ell} & \mathrm{W}_{\ell\ell} & \mathrm{W}_{\ell m} \\ \mathrm{V}^*_{km} & \mathrm{W}^*_{\ell m} & \mathrm{V}_{mm} \end{bmatrix} := \begin{bmatrix} \mathrm{v}_k \\ \mathrm{w}_\ell \\ \mathrm{v}_m \end{bmatrix} \begin{bmatrix} \mathrm{v}_k \\ \mathrm{w}_\ell \\ \mathrm{v}_m \end{bmatrix}^H && \forall \ell = (k,m) \in \mathcal{T}, \\
\xi_k &:= u_k \mathrm{V}_{kk} \in \{0, \mathrm{V}_{kk}\} && \forall k \in \mathcal{U}.
\end{aligned}
$$

The *semidefinite relaxation 1* (SDR1) is obtained by

- linearizing the power balances and branch flows using these new variables,
- relaxing V and $\mathrm{W}_{\{\ell\}}$ to positive semidefinite (instead of rank-one), and
- adding appropriate bounds on the new variables.

Tight-and-Cheap Relaxation 1 (TCR1)

TCR1 is obtained by replacing $\mathrm{V} \succeq 0$ and $\mathrm{W}_{\{\ell\}} \succeq 0$ for all $\ell \in \mathcal{T}$ in SDR1 by

$$\begin{bmatrix} 1 & \mathrm{v}_k^* & \mathrm{v}_m^* \\ \mathrm{v}_k & \mathrm{V}_{kk} & \mathrm{V}_{km} \\ \mathrm{v}_m & \mathrm{V}_{km}^* & \mathrm{V}_{mm} \end{bmatrix} \succeq 0 \qquad \forall \ell = (k, m) \in \mathcal{L} \setminus \mathcal{T},$$

$$\begin{bmatrix} 1 & \mathbf{w}_{\{\ell\}}^H \\ \mathbf{w}_{\{\ell\}} & \mathrm{W}_{\{\ell\}} \end{bmatrix} \succeq 0 \qquad \forall \ell = (k, m) \in \mathcal{T},$$

where $\mathbf{w}_{\{\ell\}}^H = (\mathrm{v}_k^*, \mathrm{w}_\ell^*, \mathrm{v}_m^*)$ for all $\ell = (k, m) \in \mathcal{T}$, and by adding the RLT constraints

$$\mathrm{Re}(\mathrm{v}_1) \geq \frac{\mathrm{V}_{11} + \underline{v}_1 \overline{v}_1}{\underline{v}_1 + \overline{v}_1},$$

$$\mathrm{Im}(\mathrm{v}_1) = 0,$$

corresponding to the reference bus $k = 1$.

Semidefinite Relaxation 2 (SDR2)

For all $\ell = (k, m) \in \mathcal{T}$, the variable definitions

$$\mathrm{W}_{k\ell} = \frac{\mathrm{V}_{kk}}{t_\ell} \quad \text{and} \quad \mathrm{W}_{\ell\ell} = \frac{\mathrm{V}_{kk}}{t_\ell^2} \qquad\qquad \forall \ell = (k, m) \in \mathcal{T},$$

are of the form $z_n = x/y^n$, $n = 1, 2$. Consider the set

$$\mathcal{S}_1 = \{(x, y, z_1, z_2) \in \mathbb{R}^4 \colon \underline{x} \le x \le \overline{x}, \underline{y} \le y \le \overline{y}, z_1 = x/y, z_2 = x/y^2\},$$

where $0 < \underline{x} < \overline{x}$ and $0 < \underline{y} < \overline{y}$. We can show that $\mathcal{S}_1$ is equivalent to

$$\mathcal{S}_2 = \{(x, z_1, z_2) \in \mathbb{R}^3 \colon z_2 = z_1^2/x, (x, z_1) \in \Omega\},$$

where $\Omega = \{(x, z_1) \in \mathbb{R}^2 \colon \underline{x} \le x \le \overline{x}, x/\overline{y} \le z_1 \le x/\underline{y}\}$ is the convex quadrilateral with vertices $(\underline{x}, \underline{x}/\overline{y})$, $(\overline{x}, \overline{x}/\overline{y})$, $(\overline{x}, \overline{x}/\underline{y})$, and $(\underline{x}, \underline{x}/\underline{y})$.

Bingane et al. (2019) propose a tighter convex set containing $\mathcal{S}_2$.

Convex Hull Representation

Proposition

The convex hull of

$$\mathcal{S}_2 = \{(x, z_1, z_2) \in \mathbb{R}^3 : \underline{x} \le x \le \overline{x}, x/\overline{y} \le z_1 \le x/\underline{y}, z_2 = z_1^2/x\},$$

where $0 < \underline{x} < \overline{x}$, *is*

$$\overline{\mathcal{S}}_2 = \{(x, z_1, z_2) \in \mathbb{R}^3 : \underline{x} \le x \le \overline{x}, z_1^2 \le x z_2, x + \underline{y}\overline{y} z_2 \le (\underline{y} + \overline{y}) z_1\}.$$

Using the Convex Hull Representation

Applying the Proposition to

$$\mathrm{W}_{k\ell} = \frac{\mathrm{V}_{kk}}{t_\ell} \quad \text{and} \quad \mathrm{W}_{\ell\ell} = \frac{\mathrm{V}_{kk}}{t_\ell^2} \qquad\qquad \forall \ell = (k,m) \in \mathcal{T},$$

we replace

$$\mathrm{W}_{k\ell} \geq \frac{\mathrm{V}_{kk}}{\bar{t}_\ell} \quad \text{and} \quad \mathrm{W}_{\ell\ell} \leq \frac{\mathrm{V}_{kk}}{\underline{t}_\ell^2} \qquad\qquad \forall \ell = (k,m) \in \mathcal{T}$$

by

$$\mathrm{V}_{kk} + \underline{t}_\ell \bar{t}_\ell \mathrm{W}_{\ell\ell} \leq (\underline{t}_\ell + \bar{t}_\ell)\mathrm{W}_{k\ell} \qquad \forall \ell = (k,m) \in \mathcal{T},$$

and correspondingly define new relaxations SDR2 and TCR2.

From the Proposition, it follows that SDR2 (respectively TCR2) is stronger than SDR1 (TCR1).

Proposed Approach to the ROPF

1. Solve SDR1, TCR1, SDR2 or TCR2 and find corresponding shunt solutions $\hat{\boldsymbol{u}} \in [0, 1]^{|\mathcal{U}|}$ and tap ratios $\hat{\boldsymbol{t}} \in \prod_{\ell \in \mathcal{T}} [\underline{t}_\ell, \bar{t}_\ell]$ with the roundoff formulas:
$$\hat{u}_k = \frac{\xi_k}{\mathrm{V}_{kk}} \in [0, 1],$$
$$\hat{t}_\ell = \sqrt{\frac{\mathrm{V}_{kk}}{\mathrm{W}_{\ell\ell}}} \in [\underline{t}_\ell, \bar{t}_\ell].$$
2. Round-off $\boldsymbol{u}$ and $\boldsymbol{t}$ to their respective nearest discrete values $\tilde{\boldsymbol{u}} \in \{0, 1\}^{|\mathcal{U}|}$ and $\tilde{\boldsymbol{t}} \in \prod_{\ell \in \mathcal{T}} \{\underline{t}_\ell, \ldots, \bar{t}_\ell\}$,
3. Fix $\boldsymbol{u} = \tilde{\boldsymbol{u}}$ and $\boldsymbol{t} = \tilde{\boldsymbol{t}}$ and solve the resulting ACOPF problem with a nonlinear (local) solver.

Summary of Computational Results

- Final solutions obtained from SDR1's and TCR2's optimal solutions have, on average, the same optimality gaps.
- Final solutions obtained from TCR1's optimal solutions have slightly larger optimality gaps.
- Solving SDR1 (respectively TCR1) is as expensive as solving SDR2 (respectively TCR2) in general.
- Solving SDR1 (respectively SDR2) is much more expensive than solving TCR1 (respectively TCR2) for large-scale instances. The TCRs are generally one order of magnitude faster than the SDRs.

Overall,
TCR2 offers the best combination of optimality gap and computation time.

Three Practical Constraints for ROPF

Limiting the number of shunts that can be switched on:

$$\sum_{n \in S} u_n \leq k. \qquad \text{(MAXkshunts)}$$

Three Practical Constraints for ROPF

Limiting the number of shunts that can be switched on:

$$\sum_{n \in S} u_n \leq k. \tag{MAXkshunts}$$

Limiting the number of switches compared to the given initial states:

$$\sum_{n \in S : u_n^0 = 0} u_n + \sum_{n \in S : u_n^0 = 1} (1 - u_n) \leq k. \tag{MAXkmoves}$$

Three Practical Constraints for ROPF

Limiting the number of shunts that can be switched on:

$$\sum_{n \in S} u_n \leq k. \qquad \text{(MAXkshunts)}$$

Limiting the number of switches compared to the given initial states:

$$\sum_{n \in S : u_n^0 = 0} u_n + \sum_{n \in S : u_n^0 = 1} (1 - u_n) \leq k. \qquad \text{(MAXkmoves)}$$

Simulating primary frequency control:

$$\begin{cases} Re(S_n) = \left[P_n^{min} + 2(P_n^0 - P_n^{min})\lambda^-\right]\delta^- \\ \qquad + \left[2P_n^0 - P_n^{max} + 2(P_n^{max} - P_n^0)\lambda^+\right]\delta^+, \quad \forall n \in G \\ 0 \leq \lambda^- \leq 0.5 \\ 0.5 \leq \lambda^+ \leq 1 \\ \lambda^-, \lambda^+ \in \mathbb{R} \\ \delta^+ + \delta^- = 1 \\ \delta^+, \delta^- \in \{0, 1\}. \end{cases} \qquad \text{(GENmoves)}$$

Simulating Primary Frequency Control

Let us look more closely at the constraints (GENmoves):

$$Re(S_n) = \left[P_n^{min} + 2(P_n^0 - P_n^{min})\lambda^-\right]\delta^- + \left[2P_n^0 - P_n^{max} + 2(P_n^{max} - P_n^0)\lambda^+\right]\delta^+, \ \forall n$$
$$0 \le \lambda^- \le 0.5, \quad 0.5 \le \lambda^+ \le 1, \quad \lambda^-, \lambda^+ \in \mathbb{R}$$
$$\delta^+ + \delta^- = 1, \quad \delta^+, \delta^- \in \{0, 1\}.$$

- (GENmoves) simulate primary frequency control by allowing active generation to increase ($\delta^+ = 1$) or decrease ($\delta^- = 1$) uniformly with respect to a given plan P^0.
- This models the fact that the generators should all move in the same direction to compensate for a lack or excess of generation.
- In addition, they must all contribute in a uniform manner according to their power bounds. This is achieved using the variables λ^+ and λ^- that make all the generators reach their upper/lower bound at the same time.
- Consequently, (GENmoves) make the constraints $P_n^{min} \le Re(S_n^{gen}) \le P_n^{max}$ redundant.

THE UNIVERSITY of EDINBURGH

Semidefinite relaxation

- Continuous part: Use of Hermitian matrix $V = vv^H$ and rank relaxation
- Combinatorial part: Introduction of variables $\xi_i = u_i V_{ii}$ and McCormick envelopes to relax quadratic constraints:

$$u_i \in [0, 1]$$
$$V_{ii} \in [(v_i^{min})^2, (v_i^{max})^2]$$

McCormick inequalities:

$$\xi_i \leq V_{ii} + (v_i^{min})^2(u_i - 1)$$
$$\xi_i \leq (v_i^{max})^2 u_i$$
$$\xi_i \geq (v_i^{max})^2(u_i - 1) + V_{ii}$$
$$\xi_i \geq (v_i^{min})^2 u_i$$

Branch-and-Bound Algorithm (B&B)

- Semidefinite relaxation solved at each node with MOSEK
- Clique decomposition algorithm of Sliwak et al. (2021) to speed up the solution of the relaxations
- Feasible solution computed at nodes with Knitro (MPEC option) after solving the semidefinite relaxation to get a good starting point
- Branching on binary variables only (i.e., no spatial B&B) $\Rightarrow$ method not guaranteed to achieve global optimality: even if all binary variables are fixed, there may still be a gap between the upper and lower bounds (if the semidefinite relaxation is not exact or if the feasible solution is not a global optimum).
- Exploration strategy: depth-first search
- Branching strategy: variable closest to 1
- Time limit of 3600 sec and fixing of some binary variables from solution of initial relaxation:
  - (MAXkshunts): fixing at 0 of binaries ≤ 0.25
  - (MAXkmoves): fixing at 1 of binaries ≥ 0.75
  - (GENmoves): fixing at 1 of binaries ≥ 0.9 and at 0 of binaries $\leq 10^{-4}$

Computational Results for (MAXkshunts)

If infeasible for $k = 4$ then k increased until feasibility is achieved

Instance	$\|S\|$	k	Root Gap	B&B #free b's	B&B #nodes	B&B Time (s)	B&B Gap	Rounding Gap
300	29	4	5.61%	7	13	32.51	**0.00%**	0.01%
300mod	29	4	0.21%	7	13	23.28	**0.00%**	5.48%
ACTIVSg500	15	4	0.52%	5	47	102.17	0.52%	0.53%
1354pegase	1082	4	-	5	1	18.80	**0.00%**	0.01%
1888rte	45	4	-	11	181	$>$3600	0.44%	0.61%
1951rte	24	4	0.11%	6	5	70.41	**0.00%**	0.11%
ACTIVSg2000	149	4	0.10%	7	23	$>$3600	0.10%	-
2736sp	1	4	**0.00%**					**0.00%**
2737sop	5	4	0.03%	5	5	244.95	**0.00%**	**0.00%**
2746wop	6	4	**0.00%**					**0.00%**
2848rte	48	4	0.04%	7	106	$>$3600	0.04%	-
2868rte	33	4	**0.00%**					-
2869pegase	2197	12	-	14	13	434.81	**0.00%**	0.02%
3012wp	9	4	0.01%	5	5	457.80	**0.00%**	**0.00%**
3120sp	9	4	0.08%	7	38	$>$3600	0.08%	-
3375wp	9	4	**0.00%**					**0.00%**
6468rte	97	4	0.08%	8	23	$>$3600	0.08%	-
6470rte	73	4	-	6	23	$>$3600	-	-
6495rte	99	14	0.40%	17	26	$>$3600	0.40%	-
6515rte	102	66	0.29%	74	26	$>$3600	0.29%	-
9241pegase	7327	125	1.44%	165	18	$>$3600	1.44%	-
13659pegase	8754	1100	1.28%	1237	28	$>$3600	1.28%	1.28%

Computational Results for (MAXkmoves)

$k = 4$ and shunts set by rounding the solution of basic relaxation of ROPF

Instance	$\|S\|$	Root Gap	B&B #free b's	B&B #nodes	B&B Time (s)	B&B Gap	Rounding Gap
300	29	**0.00%**					**0.00%**
300mod	29	**0.00%**					**0.00%**
ACTIVSg500	15	0.72%	0	1	17.38	0.61%	0.61%
1354pegase	1082	**0.00%**					**0.00%**
1888rte	45	0.38%	10	336	$>$3600	0.36%	0.39%
1951rte	24	0.01%	5	7	70.36	**0.00%**	0.01%
ACTIVSg2000	149	0.04%	54	17	1512.15	0.03%	0.03%
2736sp	1	**0.00%**					**0.00%**
2737sop	5	0.03%	1	1	57.35	**0.00%**	**0.00%**
2746wop	6	**0.00%**					**0.00%**
2848rte	48	0.03%	27	62	$>$3600	0.03%	-
2868rte	33	**0.00%**					-
2869pegase	2197	**0.00%**					**0.00%**
3012wp	9	**0.00%**					-
3120sp	9	-	2	5	829.46	0.09%	-
3375wp	9	**0.00%**					-
6468rte	97	-	41	20	$>$3600	0.03%	-
6470rte	73	-	26	14	3417.24	**0.00%**	-
6495rte	99	-	42	15	$>$3600	0.39%	-
6515rte	102	0.30%	30	18	$>$3600	0.30%	-
9241pegase	7327	1.46%	1549	12	$>$3600	1.46%	-
13659pegase	8754	-	2283	10	$>$3600	1.27%	-

Computational results - Constraint (GENmoves)

$$\begin{cases} Re(S_n) = \left[P_n^{min} + 2(P_n^0 - P_n^{min})\lambda^-\right]\delta^- \\ \qquad + \left[2P_n^0 - P_n^{max} + 2(P_n^{max} - P_n^0)\lambda^+\right]\delta^+, \ \forall n \in G \\ \delta^+ + \delta^- = 1, \ \delta^+, \delta^- \in \{0, 1\} \\ \lambda^- \in [0, 0.5], \lambda^+ \in [0.5, 1]. \end{cases} \quad \text{(GENmoves)}$$

- Omission of the thermal limits for these tests.
- Random generation of active generation plans P^0 such that total generation $\geq 1.02 \times$ total load (to allow for losses)
- Solution in two steps: $\delta^+ = 1$ and $\delta^- = 1$

Summary of results for 15 instances (with > 1000 buses) and 5 scenarios per instance, 75 cases in total:

- Rounding solved 31 cases to $< 0.05\%$ gap.
- Root/B&B solved 46 cases to $< 0.05\%$ gap.
- Root/B&B more robust than rounding: 30 cases without solution with rounding method (zero for root/B&B)

Summary and Research Challenges

Summary and Research Challenges

- Semidefinite optimization can find global optimal solutions for realistic ACOPF and ROPF instances with up to thousands of nodes.
- The computational times are still high for many practical applications.

Research Challenges

1. Gain a better understanding of chordal extensions and exploiting structure in SDRs of ACOPF.
2. Improved use of polynomial optimization and the Lasserre hierarchy.
3. Mixed-Integer Semidefinite Optimization, particularly for ROPF.
4. Uncertainty in power injections due to renewable generation and/or to deferrable load and storage devices.
5. Look at the practical aspects raised in the ARPA-E Competition!

Summary and Research Challenges

- Semidefinite optimization can find global optimal solutions for realistic ACOPF and ROPF instances with up to thousands of nodes.
- The computational times are still high for many practical applications.

Research Challenges

1. Gain a better understanding of chordal extensions and exploiting structure in SDRs of ACOPF.
2. Improved use of polynomial optimization and the Lasserre hierarchy.
3. Mixed-Integer Semidefinite Optimization, particularly for ROPF.
4. Uncertainty in power injections due to renewable generation and/or to deferrable load and storage devices.
5. Look at the practical aspects raised in the ARPA-E Competition!

You are welcome to contact me or to visit the website:

miguel.f.anjos AT ed.ac.uk `https://miguelanjos.com`

Thank you for your attention.

THE UNIVERSITY of EDINBURGH

References

- C Bingane, MF Anjos, S Le Digabel. Tight-and-cheap conic relaxation for the AC optimal power flow problem. IEEE Transactions on Power Systems 33 (6), 7181, 2018.
- C Bingane, MF Anjos, S Le Digabel. Tight-and-cheap conic relaxation for the optimal reactive power dispatch problem. IEEE Transactions on Power Systems 34 (6), 4684, 2019.
- J Sliwak, M Ruiz, MF Anjos, L Létocart, E Traversi. A Julia module for polynomial optimization with complex variables applied to optimal power flow. 2019 IEEE Milan PowerTech.
- J Sliwak, ED Andersen, MF Anjos, L Létocart, E Traversi. A Clique Merging Algorithm to Solve Semidefinite Relaxations of Optimal Power Flow Problems. IEEE Transactions on Power Systems 36 (2), 1641, 2020.
- J. Sliwak, MF Anjos, L Létocart, E Traversi. A Semidefinite Optimization-based Branch-and-Bound Algorithm for Several Reactive Optimal Power Flow Problems. arXiv preprint arXiv:2103.13648, 2021.