On chance constrained optimization

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Outline of the talk

1. **Introduction to chance constraints**
2. State-of-the art
   - Theory, Convexity, Optimality Conditions, Approximations
3. SOCP approximation of joint chance constraints
4. SOCP approximation of rectangular joint chance constraints
5. SOCP Chance constrained geometric optimization
6. SOCP Chance constrained games
7. Dependance of joint chance constraints
8. Conclusions

Problem formulation

Deterministic optimization problem:

$$\min c(x) \quad \text{s.t.} \quad g_j(x) \geq 0, j \in \mathcal{J}, x \in X_0,$$

$x \in \mathbb{R}^n$ ... decision vector
$X_0 \subset \mathbb{R}^n$ ... deterministic constraints
$\mathcal{J}$ ... index set for constraints g_j

Stochastic optimization problem:

$$\min \mathbb{E}\{c(x,\xi)\} \quad \text{s.t.} \quad G(x) = \mathbb{E}\{g_j(x;\xi)\} \geq 0, j \in \mathcal{J}, x \in X_0,$$

$\xi \in \mathbb{R}^s$ ... random vector

Constraints with a high variability: **Chance constraints**

$$\min c(x) \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_j(x;\xi) \geq 0, \} \geq 1 - \varepsilon, j \in \mathcal{J}, x \in X_0,$$

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Portfolio optimization -*A. Shapiro et al.*-

Objective: Invest a capital in such way that the expected value is maximized **under the condition that the chance of losing no more that a given a fraction** η is at least p.

Variables and parameters

$x \in \mathbb{R}^n$	... fraction of a capital invested in n instruments
$\xi_1, \ldots, \xi_n$	... random return rates in the next period
$p \in (0, 1)$	... fixed probability
$\eta = -0.1$	... protection against losses larger than 10%
$\mathbb{P}\{\sum_{i=1}^n \xi_i x_i \geq \eta\} \geq p$	... Value-at-Risk ($V@R$) constraint.

$$\max \sum_{i=1}^{n} \mathbb{E}[\xi_i] x_i \quad \text{s.t.} \quad \mathbb{P}\{\sum_{i=1}^{n} \xi_i x_i \geq \eta\} \geq p, \sum_{i=1}^{n} x_i = 1, x \geq 0.$$

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Important special cases:

1. random right-hand side only (separable case, linear or non-linear)

$$G(x) = \mathbb{P}\{g_j(x) \geq \xi, j \in \mathcal{J}\} \geq 1 - \varepsilon,$$

   i.e. $g_j(x;\xi) = g_j(x) - \xi$

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$$G(x) = \mathbb{P}\{\Xi x \leq b\}$$

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General chance constrained optimization problem

$$\min c(x) \quad \text{s.t.} \quad G(x) = \mathbb{P}\{g_k(x;\xi) \geq 0,\ k \in \mathcal{K}\} \geq p,\ x \in X_0$$

For simplicity, we assume

$$X_0 = \mathbb{R}^n \qquad c(x) = c^T x \qquad \mathcal{K} = \{1, \ldots, K\}.$$

Very classical result

Proposition 1

The feasible set is convex $\Leftrightarrow$ $G(x)$ *is quasi-concave (on* X_0*).*

We will concentrate in two important special cases

1. nonlinear constraints with random right-hand side
2. linear constraints with random coefficient matrix

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i. e.

$$g_k(x;\xi) = g_k(x) - \xi_k$$
$$G(x) = \mathbb{P}\{g_k(x) \geq \xi,\ k \in \mathcal{K}\} \geq p,$$

Denote

$$M(p) := \{x \mid \mathbb{P}\{g_k(x) \geq \xi_k,\ k \in \mathcal{K}\} \geq p\}$$

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$$\begin{aligned} \xi &= ({\Xi_1}^T, \ldots, {\Xi_k}^T) \\ g_k(x;\xi) &= h_k - \Xi x \\ G(x) &= \mathbb{P}\{\Xi_k x \leq h_k,\ k \in \mathcal{K}\} \geq p, \end{aligned}$$

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Probability function structural properties

$$G(x) := \mathbb{P}\{g(x,\xi) \geq 0\}$$

and the induced feasible set

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... are important for solving chance constrained problems.

Proposition 2 (Upper semicontinuity, closedness)

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Let ξ be an k-dimensional random vector with distribution function F_ξ

Proposition 3

if ξ has a density f_ξ, i.e., $F_\xi(z) = \int_{-\infty}^{z} f_\xi(x)dx$, then F_ξ is continuous.

Theorem 4 (Wang 1985, Romisch/Schultz 1993)

If ξ has a density f_ξ, then F_ξ is Lipschitz continuous if and only if all marginal densities f_{ξ_i} are essentially bounded.

Theorem 5 (Henrion/Romisch 2010)

If ξ has a density f_ξ such that $f_\xi^{\frac{-1}{s}}$ is convex, then F_ξ is Lipschitz continuous.

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Let $z \in \mathbb{R}^k$ be given. If all one-dimensional functions

$$\varphi^{(i)}(t) := \int_{-\infty}^{z_1} \cdots \int_{-\infty}^{z_{i-1}} \int_{-\infty}^{z_{i+1}} \cdots \int_{-\infty}^{z_k} f_\xi(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_k) du_1, \ldots, du_{i-1}, du_{i+1}, \ldots, du_k$$

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$$\mathbb{P}\{g(x) \geq \xi\} \geq p \Longleftrightarrow F_\xi(g(x)) \geq p$$

Question: Is $F_\xi \circ g$ concave ? Convex optimization methods

It suffices:

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- A nonnegative function f defined on a convex set $A \subset \mathbb{R}^m$ is called **r-concave** if $\forall (x, y) \in A$ s.t. $f(x) > 0, f(y) > 0,$ and $0 < \lambda < 1$, we have

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- A real valued function f defined on a convex set $A \subset \mathbb{R}^m$ is called **quasiconcave** if $\forall (x, y) \in A,$ and $0 < \lambda < 1$, we have

$$f(\lambda x + (1 - \lambda)y) \geq min[f(x), f(y)]$$

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Definition 7

$f(\cdot)$ is r-concave iif for any x, y such that $f(x) > 0, f(y) > 0$ and $\lambda \in [0; 1]$ we have

$$f(\lambda x + (1-\lambda)y) \geq [\lambda f^r(x) + (1-\lambda) f^r(y)]^{1/r}$$

cases $r = -\infty, 0, +\infty$ treated by continuity.

$r = -\infty$	$RHS = \min\{f(x), f(y)\}$	... f quasi-concave
$r < 0$		... f^r convex
$r = 0$	$RHS = f^\lambda(x) f^{1-\lambda}(y)\}$	... f log-concave ($\log f$ concave)
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$r = 1$		... f concave
$r = +\infty$	$RHS = \max\{f(x), f(y)\}$	... f quasi-convex

If f is r^*-concave, it is also r-concave for all $r \leq r^*$
$\Rightarrow$ every r-concave function is quasi-concave.

Prékopa(1971) (log-concave measures)

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Definition 8

Probability measure $\mathbb{P}$ (on $\mathcal{B}(\mathbb{R}^s)$) is r-concave iif for any Borel convex subset A, B such that $\mathbb{P}(A) > 0, \mathbb{P}(B) > 0$ and $\lambda \in [0; 1]$ we have

$$\mathbb{P}(\lambda A + (1-\lambda)B) \geq \left[\lambda \mathbb{P}^r(A) + (1-\lambda)\mathbb{P}^r(B)\right]^{1/r}$$

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Properties:

1. r-concave probability measure has r-concave distribution function
2. quasi-concave measure $\mathbb{P}$ on $\mathbb{R}^s$ with $\dim \operatorname{supp} \mathbb{P} = s$ has a density
3. r-concave density $\Leftrightarrow$ $\frac{r}{1+mr}$-concave measure on convex subset $\Omega \subset \mathbb{R}^s$ of dim $m > 0$ (so for $r > -\frac{1}{m}$)

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Theorem 9 (Prékopa(1995))

If

1. $g_j(x;\xi), j \in \mathcal{J}$ ... *quasi-concave functions of* x *and* ξ;
2. ξ ... *r.v. with* r*-concave density;*
3. $r \geq -\frac{1}{s}$ *(*s *is the dimension of* ξ*);*

Then

$$G(x) := \mathbb{P}\{g_j(x;\xi) \geq 0, j \in \mathcal{J}\}$$

is $\gamma = \frac{r}{1+rs}$*-concave function on the set*

$$D := \{x \mid \exists z \in \mathbb{R}^s : g_j(x;z) \geq 0 \; \forall j \in \mathcal{J}\}$$

- (ii) implies ξ have γ-concave probability
- $r = 0$ ... $G(x)$ is log-concave
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1. $g_j(\cdot,\cdot), j \in \mathcal{J}$ ... *quasi concave jointly in both arguments*
2. $\xi \in \mathbb{R}^s$ ... *random variable with r-concave probability distribution*

then

$$X_\varepsilon = \{x \mid G(x) \geq 1 - \varepsilon\}$$

is convex and closed.

- in fact, (ii) means quasi-concave probability distribution (or equivalently $-\frac{1}{s}$-concave density);
- many prominent multivariate distributions have log-concave (or at least quasi-concave and/or for some parameters) distribution (uniform, normal, Wishart, Beta, Dirichlet, Gamma, Cauchy, Pareto)

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Outline of the talk

1. Introduction to chance constraints
2. **State-of-the art**
   - Theory, Convexity, Optimality Conditions, Approximations
3. SOCP approximation of joint chance constraints
4. SOCP approximation of rectangular joint chance constraints
5. SOCP Chance constrained geometric optimization
6. SOCP Chance constrained games
7. Dependance of joint chance constraints
8. Conclusions

Literature review

Theory

- Charnes, Cooper and Symonds(1958) formulated a model with individual probabilistic constraints.

- Kataoka(1963), van de Panne and Popp(1963) proposed a solution approach for individual normal constraints.

- Miller and Wagner(1965):
  1. Investigated the model of joint constraints with **independent rvs on RHS**;
  2. They found convexity conditions using statistical hazard function,
  3. Designed three algorithms based on linearization of logarithmic chance constraints.

- Jagannathan(1974) relaxed the assumption of independence for the RHS case, and considered also the case of an independent random coefficient matrix (with a common row variance).

Andras Prekopa

- We owe Prekopa several main contributions in chance constrained optimization
  1. He studied joint constraints with dependency Prékopa(1970), and introduced quasi-concave constraint function, and log-concave measures in Prékopa(1971).
  2. He proved the convexity for right-hand sided problems for log-concave distribution and concave g_i in Prékopa(1971) .
  3. He studied the property of α-concavity for many prominent distributions (multivariate gamma) in Prékopa and Szántai(1978).
  4. He gave the generalized definition of α-concave measures on a set, and used for extending optimality and duality theory in Dentcheva, Prekopa and Ruszczynski(2000, 2002).
  5. He introduced p-efficient points in Prékopa(1990)

Approximations of chance constraints

ACC

- Sample average approximation (SAA) studied by Shapiro(1993), Ahmed and Shapiro(2002), . . .
- Scenario approximation method studied by Calafiore and Campi(2005), Calafiore and Campi(2006), extended by Nemirovski and Shapiro(2006).
- Robust optimization scheme: Ben-Tal and Nemirovski(1998), Bertsimas and Sim(2004), El Ghaoui and Lebret(1997)
- Integer programming approach: Luedtke and Ahmed(2008), Luedtke, Ahmed and Nemhauser(2010) . . .

Approximations of chance constraints

ACC

- *Second-order cone programming* considered by Cheng and Lisser(2012).
- *A completely positive representation of 0-1 linear programs with joint probabilistic constraints* Cheng and Lisser(2013).
- *Distributionally Robust Stochastic Knapsack Problem* Cheng, Delage and Lisser (2014).
- *Chance constrained 0-1 quadratic programs using copulas* Cheng and Lisser(2015).
- *Stochastic games* in Singh, Lisser and Jouini (2015).
- *Stochastic geometric optimization with joint probabilistic constraints* in Liu, Lisser and Chen (2016).
- *Variational inequality formulation for the games with random payoffs*, Vikas Vikram Singh, Abdel Lisser, (2018).
- *A Characterization of Nash Equilibrium for the Games with Random Payoffs*, Vikas Vikram Singh, Abdel Lisser, (2018).
- *Rectangular chance constrained geometric optimization* in Jia Liu, Shen Peng, Abdel Lisser, Zhiping Chen, (2019).
- *A second-order cone programming formulation for two player zero-sum games with chance constraints*, Vikas Vikram Singh, Abdel Lisser, (2019).

. . . Several solving methods

- Cutting plane method
- Logarithmic barrier function method
- Dual method (nonlinear probabilistic problems)
- Primal-Dual method (linear probabilistic problems)
- Nonparametric estimates of distribution functions (joint chance linear constraints)
- A response surface method
- Discrete distributions
- Probabilistic valid inequalities

See Shapiro et al (2006) for more details

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$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{\Xi x \leq h\} \geq p$$

- If the rows ${\Xi_k}^T$ are independent and follow $N_n(\mu_k; \Sigma_k \succ 0)$ then

$$\xi_k(x) := \frac{\Xi_k^T x - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}, \qquad g_k(x) := \frac{h_k - \mu_k^T x}{\sqrt{x^T \Sigma_k x}},$$

so

$$G(x) = \mathbb{P}\{g_k(x) \geq \xi_k(x), \ k \in \mathcal{K}\}$$

where $\xi_k(x) \sim N(0; 1)$.

- If $K = 1$ (only one constraint) then

$$X(p) = \left\{ x \in X \mid {\mu_1}^T x + F^{-1}(p) \sqrt{x^T \Sigma_1 x} \leq h_1 \right\}.$$

- To extend the result to $K > 1$ (more constraint rows): introduce auxiliary variables y_k. If the rows are independent, then (Cheng and Lisser(2012))

$$X(p) = \left\{ x \in X \mid \exists y_k \geq 0, \ \sum_{k=1}^{K} y_k = 1 : \right.$$

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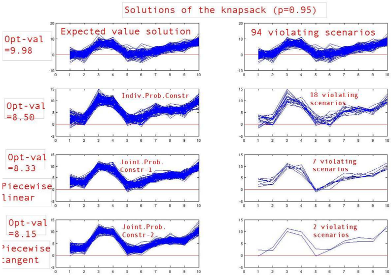
Normal distribution

Normal distribution with independent constraint rows



Problem formulation

Consider the following joint linear chance constrained problem:

$$\min c^T x \quad \text{s.t.} \quad \mathbb{P}\{T_j x \leq D_j, j \in K\} \geq p, x \in X,$$

$X \subset R^n_+$	... is a polyhedron
$c \in R^n$,	... $D = (D_1, \dots, D_k) \in R^K$
$T = [T_1, \dots, T_K]'$	... is a $K \times n$ random matrix
$T_k, k = 1, \dots, K$	... is a random vector in R^n
p	... is a confidence parameter

1. T_k is multivariate normally distributed with mean $\mu_k = (\mu_{k1}, \dots, \mu_{kn})$ and covariance matrix Σ_k.
2. Moreover, T_{k_i} and T_{k_j} are pairwise independent.

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Hence we obtain the following deterministic nonlinear problem

$$\min c^T x \quad \text{s.t.} \quad \mu_k^T x + F^{-1}(p^{y_k})||\Sigma_k^{1/2} x|| \leq D_k, k = 1, \ldots, K$$
$$\sum_{k=1}^{K} y_k = 1, \; y_k \geq 0, k = 1, \ldots, K$$
$$x \in X.$$

Consequence: linear approximation of $F^{-1}(p^{y_k})$ results in

1. piecewise tangent approximation ($\Rightarrow$ outer bound for feasible solutions)
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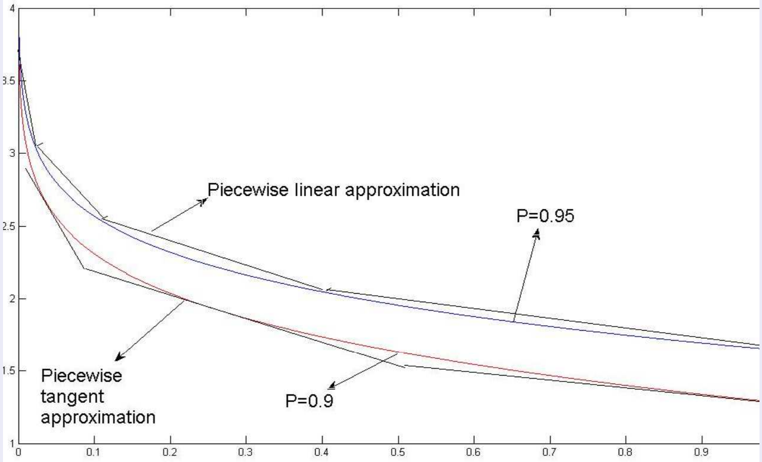
Approximations of NLPJP

Approximation of $F^{-1}(p^{y_k})$



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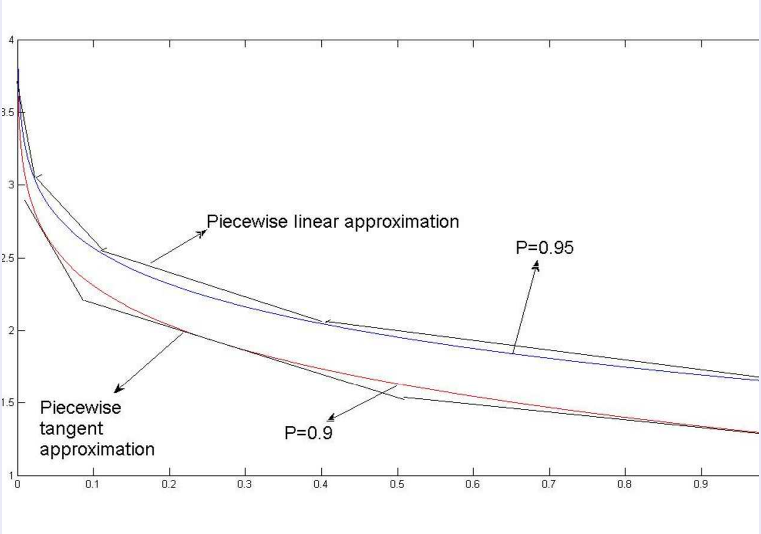

Convex approximation of the feasible set: main result

Chance-constrained problem with independent constraint rows

$$\min c^T x \quad \text{s.t.} \quad \mathbb{P}\{Tx \leq D\} \geq p$$

Theorem 11

1. *Let* $y_{ki} = y_k x_i, k = 1, \ldots, K, i = 1, \ldots, n$
2. $\tilde{z}_k = (\tilde{z}_{k1}, \ldots, \tilde{z}_{kn})$.
3. *Together with the approximation of* $F^{-1}(p^{y_k})$*, we have*

$$\begin{aligned} \min c^T x \quad s.t. \quad & \mu_k^T x + ||\Sigma_k^{1/2} \tilde{z}_k|| \leq D_k, \quad k = 1, \ldots, K, \\ & \tilde{z}_{ki} \geq a_j x_i + b_j y_{ki}, \quad j = 0, 1, \ldots, N-1, i = 1, \ldots, n \\ & \sum\nolimits_k y_{ki} = x_i, \; y_{ki} \geq 0, \; k \in K, i = 1, \ldots, n, x \in X. \end{aligned}$$

where $a_0 = 0, b_0 = 0$.
Moreover, if

3. $z_N = 1$ *and the feasible set of* $y_k, k = 1, \ldots, K$ *is bounded by* $[z_1, 1]^K$

then the optimum of the approximation is an upper bound of the deterministic equivalent problem.

Convex approximation of the feasible set: main result

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& \tilde{z}_{ki} \ge \hat{a}_j x_i + \hat{b}_j y_{ki}, \quad j = 0, 1, \dots, N-1, i = 1, \dots, n \\
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where $a_0 = 0, b_0 = 0$.
Moreover, the optimum of the approximation is a lower bound of the deterministic equivalent problem.

The Restricted shortest path (RCSP) with joint constraints can be formulated as

$$\min c(x) \quad \text{s.t.} \quad Tx \leq D,\ Mx = b,\ x \in \{0,1\}^n$$

$c \in R^n$ ... cost (deterministic) vector
$M \in \mathbb{R}^{m \times n}$ ... the *node-arc incidence matrix*
$b \in \mathbb{R}^m$ ... where all elements are 0 except the *s-th*
T ... is a non negative $K \times n$ matrix
D ... is a positive vector of K elements.

We assume that the arc used resources are pairwise normal independent random variables.

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Restricted shortest path with joint constraints

Assumptions

We assume that, for each arc of the network, the resources consumed by traversing the arc are random and independently normally distributed.

Relaxed RCSP with joint constraints

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & \boldsymbol{Pr}\{Tx \leq D\} \geq 1 - \alpha \\ & Mx = b \\ & x \geq 0 \end{aligned}$$

which is a NLPJP problem.

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Restricted shortest path with joint constraints

Assumptions

We assume that, for each arc of the network, the resources consumed by traversing the arc are random and independently normally distributed.

Relaxed RCSP with joint constraints

$$
\begin{aligned}
\min \quad & c^T x \\
s.t. \quad & \boldsymbol{Pr}\{Tx \leq D\} \geq 1 - \alpha \\
& Mx = b \\
& x \geq 0
\end{aligned}
$$

which is a NLPJP problem.

1. the problems were solved by **Sedumi**
2. all tests ran on a Pentium(R)D @ 3.00 GHz with 2.0 GB RAM
3. 5 instances taken from the OR-library with 10 constrained resources

Instances and parameters

$(100- > 200, 999- > 1960)$	... graph sizes
$[0, \sigma^2(k)]$	... means and variances interval
$z_1 = e^{-6}, z_2 = 0.15, z_3 = 1$	... piecewise linear approximation points
$z_1 = 0.15$ and $z_2 = 0.45$	... piecewise tangent approximation points
$p = 0.99$	... the confidence parameter

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$p = 0.99$	... the confidence parameter

Numerical results

Instances	(Nodes, Arcs)	Lower bound	CPU time (s)	Upper bound	CPU time (s)	Gap(%)
RCSP1	(100,990)	100.20	18.72	104.20	15.74	3.84
RCSP3	(100,999)	6.16	17.54	6.61	17.91	6.81
RCSP4	(100,999)	8.88	19.49	9.98	18.48	11.02
RCSP6	(200,1960)	5.00	24.84	5.00	28.36	0.00
RCSP7	(200,1960)	5.40	39.79	5.65	35.95	4.42

Outline of the talk

1. Introduction to chance constraints
2. State-of-the art
   - Theory, Convexity, Optimality Conditions, Approximations
3. SOCP approximation of joint chance constraints
4. **SOCP** approximation of rectangular joint chance constraints
5. SOCP Chance constrained geometric optimization
6. SOCP Chance constrained games
7. Dependance of joint chance constraints
8. Conclusions

0-1 quadratic program with chance rectangular constraints

$$
\begin{aligned}
\min x^T P_0 x + p_0^T x \quad \text{s.t.} \quad & \mathbb{P}\{Ax + D_1 \leq Tx \leq Bx + D_2\} \geq 1 - \alpha \\
& w_t^T x = d_t, \quad t = 1, \dots, m \\
& \bar{w}_{\bar{t}}^T x \leq \bar{d}_{\bar{t}}, \quad \bar{t} = 1, \dots, \bar{m} \\
& x^T P_1 x + p_1^T x \leq d_0 \\
& x \in \{0,1\}^n.
\end{aligned}
$$

where

$T = [T_1, \dots, T_K]^T$	... normally distributed random matrix
$\mu_k = (\mu_{k1}, \dots, \mu_{kn}), \Sigma_k$	... known mean and covariance
$p_0,\ p_1,\ w_t,\ \bar{w}_{\bar{t}} \in \mathbb{R}^n$	... real vectors
$D_i = (D_{i1}, \dots, D_{iK}) \in R^K,\ i = 1, 2$	... given real matrices
$P_0,\ P_1 \in R^{n \times n},\ A,\ B \in R^{K \times n}$	... deterministic matrices

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... $T_k, k = 1, \ldots, K$, are pairwise independent

$$\mathbb{P}\{Ax + D_1 \le Tx \le Bx + D_2\} \ge 1 - \alpha$$

is equivalent to

$$\prod_{k=1}^{K} \mathbb{P}\{A_k^T x + D_{1k} \le T_k^T x \le B_k^T x + D_{2k}\} \ge 1 - \alpha$$

... which is satisfied iff there exists $y \in \mathbb{R}_+^K$ s.t $\sum_{k=1}^{K} y_k = 1$, and

$$\mathbb{P}\{A_k^T x + D_{1k} \le T_k^T x \le B_k^T x + D_{2k}\} \ge (1-\alpha)^{y_k}, \; \forall k \in \{1, 2, \ldots, K\}$$

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Let z_k and $\bar{z}_k, \; k \in \{1, 2, \ldots, K\}$ be two additional auxiliary variables

... We have the following equivalent formulation

$$\mathbb{P}\{A_k^T x + D_{1k} \leq T_k^T x \leq B_k^T x + D_{2k}\} \geq (1-\alpha)^{y_k}$$

this is equivalent to

$$\mathbb{P}\{T_k^T x \leq B_k^T x + D_{2k}\} \geq z_k$$
$$\mathbb{P}\{A_k^T x + D_{1k} \leq T_k^T x\} \geq \bar{z}_k$$
$$z_k + \bar{z}_k \geq 1 + p^{y_k}$$
$$0 \leq z_k, \bar{z}_k \leq 1$$

for $k = 1, \ldots, K$.

Deterministic formulation of the rectangular chance constrained problem

$$
\begin{aligned}
\min x^T P_0 x + p_0^T x \quad \text{s.t.} \quad & (F^{-1}(z_k))^2 x^T \Sigma_k x \le (B_k^T x + D_{2k} - \mu_k^T x)^2, \forall k \\
& (F^{-1}(\bar{z}_k))^2 x^T \Sigma_k x \le (\mu_k^T x - A_k^T x - D_{1k})^2, \forall k \\
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& \bar{w}_{\bar{t}}^T x \le \bar{d}_{\bar{t}}, \quad \forall \bar{t} \\
& \mu_k^T x \le B_k^T x + D_{2k},\ \mu_k^T x \ge A_k^T x + D_{1k}, \forall k \\
& z_k + \bar{z}_k \ge 1 + p^{y_k}, 0 \le z_k, \bar{z}_k \le 1, \forall k \\
& \sum_{k=1}^{K} y_k = 1, \quad y_k \ge 0, \forall k \\
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... As $(F^{-1}(z))^2$ and p^y are convex when $z \geq 0.5$ and $y \geq 0$ respectively, they can be approximated by a piecewise linear approximation

Lemma 13 (Chang and A.L (2015))

With the piecewise tangent approximation of $(F^{-1}(z_k))^2$, and after linearizing the quadratic terms, i.e., $Z^k_{i,j} = \bar{F}_k X_{ij}, \bar{Z}^k_{i,j} = \bar{F}_k X_{ij}, X_{ij} = x_i x_j$, we obtain an MILP.

For the sake of simplicity, we get the standard formulation as Burrer (2009) adding appropriate slack variables:

$$\begin{array}{ll} \min & v^T \bar{P}_0 v + \bar{p}_0^T v \\ s.t. & \hat{w}_t v = \hat{d}_t, \quad t = 1, \dots, M \\ & v_i \in \{0, 1\}, \quad i = 1, \dots, n \\ & v \in \mathbb{R}^N_+ \end{array} \tag{1}$$

where $\bar{P}_0$, $\bar{p}_0$, $\hat{c}$, $\hat{w}_t$ and $\hat{d}_t$, M, N are defined accordingly.

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where $\bar{P}_0$, $\bar{p}_0$, $\hat{c}$, $\hat{w}_t$ and $\hat{d}_t$, M, N are defined accordingly.

The SDP relaxation of QPJPC together with some redundant constraints to strength the quality of the bounds is given by

$$\min\langle P_0, X\rangle + p_0^T x \quad \text{s.t.} \quad \begin{array}{l} ||\Sigma_k^{1/2} v_k|| \leq B_k^T x + D_{2k} - \mu_k^T x, \forall k \\ ||\Sigma_k^{1/2} \bar{v}_k|| \leq \mu_k^T x - A_k^T x - D_{1k}, \forall k \\ \text{linearization constraints} \\ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \\ X_{ii} = x_i \geq 0, i = 1, \ldots, n. \end{array}$$

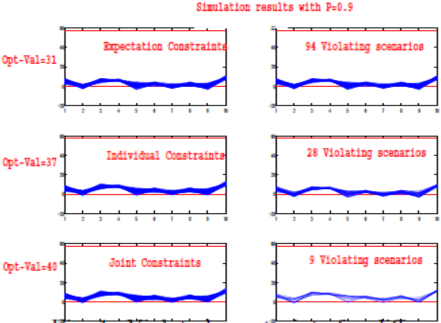

Fig. 1. Violated constraints for different models

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A geometric program can be formulated as

$$\min_{t} \quad g_0(t) \text{ s.t.} \quad g_k(t) \leq 1,\ k = 1, \cdots, K,\ t \in \mathbb{R}^M_{++}$$

with

$$g_k(t) = \sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}},\ k = 0, \cdots, K.$$

$\{I_k,\ k = 0, \cdots, K\}$ is the disjoint index sets of $\{1, \cdots, Q\}$.

We call $c_i \prod_{j=1}^{M} t_j^{a_{ij}}$ a monomial and $g_k(t)$ a posynomial.

Geometric programs (Cont'd)

Geometric programs have a number of practical problems, such as

- shape optimization problems (Boyd et al., 2007)
- electrical circuit design problems (Boyd et al., 2007)
- mechanical engineering problems (Wiebking, 1977)
- economic and managerial problems (Luptáčik, 1981)
- nonlinear network problems (Kim et al., 2007)

Stochastic geometric programs

- Generally, c_i are supposed to be non-negative coefficients.
- In real life problems, c_i is not known deterministically but randomly.
- As c_i are random variables, we formulate the problem as stochastic geometric program.
- We use probabilistic constraints to model the uncertainty in the posynomial constraints.

Stochastic geometric programs with individual probabilistic constraints can be written as:

$$
\begin{aligned}
(SGPIPC) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
& \text{s.t.} \quad P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1 - \epsilon_k, \ k = 1, \cdots, K.
\end{aligned}
$$

where $\epsilon_k \in (0, 0.5]$ is the tolerance probability for the k-th posynomial constraint.

Stochastic geometric programs (Cont'd)

Stochastic geometric programs with joint probabilistic constraints:

$$
\begin{aligned}
(SGPJPC) \quad & \min_{t \in \mathbb{R}^M_{++}} && E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
& \text{s.t.} && P\left(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1, \ k = 1, \cdots, K\right) \geq 1 - \epsilon.
\end{aligned}
$$

where $\epsilon \in (0, 0.5]$ is the tolerance probability for all the posynomial constraints.

Our work

We are interested by stochastic geometric program with joint probabilistic constraints.

- We suppose that a_{ij} is deterministic and c_i is normally distributed and parwise independent of each other, i.e., $c_i \sim N(E_{c_i}, \sigma_i^2)$.

We consider the following tools:

1. standard variable transformation from geometric programming
2. piecewise linear approximation
3. sequential convex approximation

Equivalent formulation

As c_i are paiwise independent, we have

$$P\left(\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1,\ k=1,\cdots,K\right) \geq 1-\epsilon$$

is equivalent to

$$\prod_{k=1}^{K} P\left(\sum_{i\in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\right) \geq 1-\epsilon.$$

Equivalent formulation (Cont'd)

We introduce auxiliary variables $y_k \in \mathbb{R},\ k = 1, \cdots, K$, (SGPJPC) problem can be equivalently reformulated as

$$
\begin{aligned}
\min_{t \in \mathbb{R}^M_{++}, y \in \mathbb{R}^K} \quad & E\left[\sum_{i \in I_0} c_i \prod_{j=1}^{M} t_j^{a_{ij}}\right] \\
\text{s.t.} \quad & P\Big(\sum_{i \in I_k} c_i \prod_{j=1}^{M} t_j^{a_{ij}} \leq 1\Big) \geq y_k,\ k = 1, \cdots, K, \\
& \prod_{k=1}^{K} y_k \geq 1 - \epsilon,\ y_k \geq 0.
\end{aligned}
$$

Equivalent formulation (Cont'd)

As $c_i \sim N(E_{c_i}, \sigma_i^2)$, (SGPJPC) problem is equivalent to

$$\begin{aligned}
\min_{t \in \mathbb{R}^M_{++}, y \in \mathbb{R}^K} \quad & \sum_{i \in I_0} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} \\
\text{s.t.} \quad & \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k) \sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1, \ k = 1, \cdots, K, \\
& \prod_{k=1}^{K} y_k \geq 1 - \epsilon, \ y_k \geq 0.
\end{aligned}$$

$\Phi^{-1}(y_k)$ is the quantile of standard normal distribution $N(0, 1)$.

Equivalent formulation (Cont'd)

The standard variable transformation $r_j = \log(t_j),\ j = 1, \cdots, M$ and $x_k = \log(y_k),\ k = 1, \cdots, K$ leads to the equivalent formulation:

$$\min_{r \in \mathbb{R}^M, x \in \mathbb{R}^K} \quad \sum_{i \in I_0} E_{c_i} \exp\left\{ \sum_{j=1}^{M} a_{ij} r_j \right\}$$

$$\text{s.t.} \quad \sum_{i \in I_k} E_{c_i} \exp\left\{ \sum_{j=1}^{M} a_{ij} r_j \right\} + \sqrt{\sum_{i \in I_k} \sigma_i^2 \exp\left\{ \sum_{j=1}^{M} (2a_{ij} r_j + \log(\Phi^{-1}(e^{x_k})^2)) \right\}} \leq 1,\ k = 1, \cdots, K,$$

$$\sum_{k=1}^{K} x_k \geq \log(1 - \epsilon),\ x_k \leq 0,\ k = 1, \cdots, K.$$

Theorem 14

An approximation of (SGPJPC) problem can be found by using the piecewise linear function $F(x_k)$:

$$(SGP_A)$$

$$\min_{r\in\mathbb{R}^M, x\in\mathbb{R}^K} \quad \sum_{i\in I_0} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\}$$

$$\sum_{i\in I_k} E_{c_i} \exp\left\{\sum_{j=1}^{M} a_{ij} r_j\right\} + \sqrt{\sum_{i\in I_k} \sigma_i^2 \exp\left\{\sum_{j=1}^{M} (2a_{ij} r_j + d_s x_k + b_s)\right\}} \leq 1,\ s = 1,\cdots,S,\ k = 1,\cdots,K,$$

$$\sum_{k=1}^{K} x_k \geq \log(1-\epsilon),\ x_k \leq 0,\ k = 1,\cdots,K.$$

The optimal value is a lower bound of the (SGPJPC) problem. When S goes to infinity, the approximation is tight.

Sequential convex approximation

- Sequential convex approximation $\Rightarrow$ upper bound
- Basic idea: decompose the problem into subproblems where a subset of variables is fixed alternatively.
- We first fix $y = y^n$ and update t by solving

$$
\begin{aligned}
(SQ_1) \quad & \min_{t \in \mathbb{R}^M_{++}} \quad \sum_{i \in I_0} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} \\
& \text{s.t.} \quad \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} t_j^{a_{ij}} + \Phi^{-1}(y_k^n) \sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} t_j^{2a_{ij}}} \leq 1, \\
& \qquad k = 1, \cdots, K
\end{aligned}
$$

Sequential convex approximation (Cont'd)

- and then fix $t = t^n$ and update y by solving

$$
\begin{aligned}
(SQ_2) \quad & \min_{y \in \mathbb{R}_+^K} \quad \sum_{k=1}^{K} \phi_k \; y_k \\
& \text{s.t.} \quad y_k \le \Phi\left(\frac{1 - \sum_{i \in I_k} E_{c_i} \prod_{j=1}^{M} (t_j^n)^{a_{ij}}}{\sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} (t_j^n)^{2a_{ij}}}} \right), \; k = 1, \cdots, K. \\
& \qquad \prod_{k=1}^{K} y_k \ge 1 - \epsilon, \; y_k \ge 0, \; k = 1, \cdots, K.
\end{aligned}
$$

- ϕ_k is a chosen searching direction.

Sequential convex approximation (Cont'd)

Algorithm 1 Sequential convex approximation

Initialization:

Choose an initial point y^0 of y feasible for (8). Set $n = 0$.

Iteration:

while $n \geq 1$ and $||y^{n-1} - y^n||$ is small enough **do**

- Solve problem (SQ_1); let t^n, θ^n and v^n denote an optimal solution of t, an optimal solution of the Lagrangian dual variable θ and the optimal value, respectively.
- Solve problem (SQ_2) with $\phi_k = \theta_k^n \cdot (\Phi^{-1})'(y_k^n) \sqrt{\sum_{i \in I_k} \sigma_i^2 \prod_{j=1}^{M} (t_j^n)^{2a_{ij}}}$; let $\tilde{y}$ denote an optimal solution.
- $y^{n+1} \leftarrow y^n + \tau(\tilde{y} - y^n)$, $n \leftarrow n + 1$. Here, $\tau \in (0, 1)$ is the step length.

end while

Output: t^n, v^n

Sequential convex approximation (Cont'd)

Theorem 15

Algorithm 1 converges in a finite number of iterations and the returned value v^n is a upper bound for problem (SGP).

- Problems (SQ_1) and (SQ_2) are both geometric programs, hence they can be transformed into a convex programming problem, and solved by interior point methods.

Shape optimization problem

Consider a joint probabilistic constrained shape optimization problem,

$$
\begin{aligned}
\min_{h,w,\zeta} \quad & h^{-1}w^{-1}\zeta^{-1} \\
\text{s.t.} \quad & P\Big((2/A_{wall})hw + (2/A_{wall})h\zeta \leq 1,\ (1/A_{flr})w\zeta \leq 1\Big) \geq 1-\epsilon, \\
& \alpha h^{-1}w \leq 1,\ (1/\beta)hw^{-1} \leq 1, \\
& \gamma w\zeta^{-1} \leq 1,\ (1/\delta)w^{-1}\zeta \leq 1.
\end{aligned}
$$

- maximize the volume of a box-shaped structure with height h, width w and depth ζ
- subject to the total wall area $2(hw + h\zeta)$, and floor area $w\zeta$

Settings

- Set $\alpha = \gamma = 0.5$, $\beta = \delta = 2$, $\epsilon = 5\%$,
- Assume $1/A_{wall} \sim N(0.005, 0.01)$ and $1/A_{flr} \sim N(0.01, 0.01)$.
- We use CVX software to solve the approximation problems with Matlab R2012b on a PC with a 2.6 Ghz Intel Core i7-5600U CPU and 12.0 GB RAM.
- We solve five groups of approximation problems with different number of segments, S.

Computational results

Table 1: Computational results

S	Var. Num.	Con. Num.	Low. bound	CPU(s)	Upp. bound	CPU(s)	Gap(%)
1	133	60	0.232	0.5955	0.256	5.5274	9.655
2	184	91	0.234	0.6272	0.256	5.5274	8.789
5	283	153	0.241	0.9480	0.256	5.5274	6.044
10	513	273	0.252	1.3554	0.256	5.5274	1.713
20	973	513	0.256	1.9986	0.256	5.5274	0

Sequential convex approximation algorithm converges within 7 outer iterations

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n-player game

- $I = \{1, 2, \cdots, n\}$ – set of players,
- A_i – a finite action set of Player i, and $A = \prod_{i \in I} A_i$.
- $a = (a_1, a_2, \cdots a_n) \in A$ – an action profile.
- $\tilde{r}_i(a)$ – payoff of player i.
- $\tilde{r}_i = (\tilde{r}_i(a))_{a \in A}$ – payoff vector of player i.

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- X_i – set of mixed strategies of player i.
- $X = \prod_{i \in I} X_i$ – set of all mixed strategy profiles.
- Denote a mixed strategy of player i by τ_i.
- $\tau_{-i} = (\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \cdots, \tau_n)$ is strategy profile of all other players except player i.

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Nash equilibrium

- For a given strategy profile $\tau = (\tau_1, \tau_2, \cdots, \tau_n) \in X$ the payoff of player i is defined by

$$r_i(\tau) = \sum_{a \in A} \left(\prod_{j \in I} \tau_j(a_j) \right) \tilde{r}_i(a). \quad (2)$$

- A strategy profile τ^* is said to be a Nash equilibrium if for each $i \in I$,

$$r_i(\tau_i^*, \tau_{-i}^*) \geq r_i(\tau_i, \tau_{-i}^*), \ \forall\ \tau_i \in X_i. \quad (3)$$

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The model

- We consider the games where the payoff vector $(\tilde{r}_i(a))_{a \in A}$ of player i is a random vector.

- Let Ω be a sample space. Then, for an $\omega \in \Omega$, the payoff of player i at action profile a is given by $\tilde{r}_i(a, \omega)$.

- Similarly, for an ω, payoff of player i at mixed strategy profile τ is given by

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Assumptions

- We assume that the distribution of the payoff vector of each player is known to all the players.
- α is known to all the players.
- For a given $\alpha \in [0,1]^n$, the payoff function of a player is known to all other players.
- The chance-constrained game is a non-cooperative game with complete information.

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Best response and Nash equilibrium

- The set of best response strategies is given by

$$BR_i^{\alpha_i}(\tau_{-i}) = \{\bar{\tau}_i \in X_i | u_i^{\alpha_i}(\bar{\tau}_i, \tau_{-i}) \geq u_i^{\alpha_i}(\tau_i, \tau_{-i}),\ \forall\ \tau_i \in X_i\}.$$

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$$u_i^{\alpha_i}(\tau) = \sup\{\gamma | P(r_i(\tau) \geq \gamma) \geq \alpha_i\}.$$

- $r_i(\tau)$ is defined as

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CCG

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Nash Existence Theorem

Theorem 16

Consider an n-player game where each player has finite number of actions. If the payoff vector $(\tilde{r}_i(a))_{a \in A}$ of player i, $i \in I$, follows a multivariate elliptically symmetric distribution with location parameter $\mu_i = (\mu_i(a))_{a \in A}$ and scale matrix Σ_i which is positive definite, then there exists a mixed strategy Nash equilibrium for all $\alpha \in (0.5, 1]^n$.

Vikas Vikram Singh, Oualid Jouini, and Abdel Lisser, *Existence of Nash equilibrium for chance-constrained games*, **Operations Research Letters**, *44 (5): 640–644, 2016.*

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Dependence

- Many practical results concerning chance-constrained programming is based on the assumption of **independence** of random rows of the problem.
- The proof are usually based on the separability property of the form

$$F(\xi) = \prod_{k \in \mathcal{K}} F_k(\xi_k)$$

where F is the distribution function of ξ and F_k its marginals.

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Definition 17

The copula is the distribution function $C : [0;1]^K \to [0;1]$ of some K-dimensional random vector whose marginals are uniformly distributed on $[0;1]$.

Proposition 18 (Sklar's theorem)

For any K-dimensional distribution function $F : \mathbb{R}^K \to [0;1]$ with marginals $F_1, \dots, F_K$, there exists a copula C such that

$$\forall z \in \mathbb{R}^K \quad F(z) = C\big(F_1(z_1), \dots, F_K(z_K)\big).$$

If, moreover, F_k are continuous, then C is uniquely given by

$$C(u) = F\left(F_1^{-1}(u_1), \dots, F_K^{-1}(u_K)\right).$$

Otherwise, C is uniquely determined on $\operatorname{range} F_1 \times \cdots \times \operatorname{range} F_K$.

Definition 19

A copula C is called Archimedean if there exists a continuous strictly decreasing function $\psi : [0;1] \to \mathbb{R}_+$, called generator of C, such that $\psi(1) = 0$ and

$$C(u) = \psi^{-1}\left(\sum_{i=1}^{n} \psi(u_i)\right).$$

If $\lim_{u\to 0} \psi(u) = +\infty$ then C is called a strict Archimedean copula and ψ is called a strict generator.

Properties of copula generator

- ψ is convex
- ψ^{-1} is continuous strictly decreasing convex on $[0;\psi(0)]$

Dependence: copulae

Archimedean copulae: copula generators

Practical advantages of Archimedean copulae:

- associative class (separability)
- explicit formulae of generators for most common copulae
- model dependence through one "nice" generator function and usually one or a small number of parameters
- many families adapted to a concrete problem setting (Nelsen(2006) provides a table of 22 one-parameter families of Archimedean copulae)

Some known shortcomings:

- Gaussian copula is not Archimedean
- all copula margins are the same ⇒ only symmetric dependence structures described
- limited area of dependence structures (due to small number of parameters)
- less ability to capture negative dependence

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1. independence (product) copula

$$\psi(t) := -\ln t,$$

2. Gumbel-Hougaard copulae, with $\theta \geq 1$

$$\psi_\theta(u) := (-\ln t)^\theta$$

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$$\min c^T x \quad \text{s.t.} \quad G(x) = \mathbb{P}\{\Xi x \leq h\} \geq p$$

- If the rows ${\Xi_k}^T$ are independent and follow $N_n(\mu_k; \Sigma_k \succ 0)$ then

$$\xi_k(x) := \frac{\Xi_k^T x - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}, \qquad g_k(x) := \frac{h_k - \mu_k^T x}{\sqrt{x^T \Sigma_k x}},$$

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$$G(x) = \mathbb{P}\{g_k(x) \geq \xi_k(x),\ k \in \mathcal{K}\}$$

where $\xi_k(x) \sim N(0;1)$.

- If $K = 1$ (only one constraint) then

$$X(p) = \left\{ x \in X \mid {\mu_1}^T x + \Phi^{-1}(p) \sqrt{x^T \Sigma_1 x} \leq h_1 \right\}.$$

- To extend the result to $K > 1$ (more constraint rows): introduce auxiliary variables y_k. If the rows are independent, then (Cheng and Lisser(2012))

$$X(p) = \left\{ x \in X \mid \exists y_k \geq 0,\ \sum_{k=1}^{K} y_k = 1 : \right.$$

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Normal distribution

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- If $K > 1$ with dependent random vectors then (Cheng, Houda and Lisser(2014)

$$X(p) = \Bigg\{x \in X \mid \exists y_k \geq 0,\ \sum_{k=1}^{K} y_k = 1 :$$
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$$\min x^T Qx + c^T x \quad \text{subject to} \quad \mathbb{P}\{Tx \leq d\} \geq p,\ Ax = b,\ x \in \{0,1\}^n$$

where

1. $c \in \mathbb{R}^n$, $d \in \mathbb{R}^K$, and $b \in \mathbb{R}^m$ are deterministic vectors,
2. $Q \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{m\times n}$ are deterministic matrices,
3. $T \in \mathbb{R}^{K\times n}$ is a random matrix with rows $T_k^T,\ k = 1, \ldots, K$,
4. $p \in (0;1)$ is a prescribed probability level.

Lemma 20

If the random vector $(\xi_1(x), \ldots, \xi_K(x))^T$, where $\xi_k(x)$ has one-dimensional standard normal distribution, has a joint distribution driven by the Gumbel-Hougaard copula C_θ with some $\theta \geq 1$ then the constraint $\mathbb{P}\{Tx \leq d\} \geq p$ is equivalent to the set of constraints

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$$\sum_k z_k = 1, \ z_k \geq 0 \ \forall k$$

where $\Phi(\cdot)$ is the inverse of the standard normal cumulative distribution function.

Lemma 21

If $p \geq \frac{1}{2}$ and $\theta \geq 1$ then $H(z) := \Phi^{-1}\left(p^{z^{1/\theta}}\right)$ is convex on $[0; 1]$.

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Consequence: deterministic reformulation given by

$$\min x^T Qx + c^T x \quad \text{subject to} \quad \mu_k^T x + \Phi^{-1}\left(p^{z_k^{1/\theta}}\right)\|\Sigma_k^{1/2}x\| \le d_k \ \forall k,$$
$$\sum_{k=1}^{K} z_k = 1,\ z_k \ge 0\ \forall k,\ Ax = b,\ x \in \{0,1\}^n. \tag{4}$$

The problem (4) is equivalent to

$$\min x^T Qx + c^T x \quad \text{subject to} \quad \left(\Phi^{-1}\left(p^{z_k^{1/\theta}}\right)\right)^2 x^T \Sigma_k x \le (d_k - \mu_k^T x)^2 \ \forall k,$$
$$\mu_k^T x \le d_k \ \forall k,$$
$$\sum_{k=1}^{K} z_k = 1,\ z_k \ge 0\ \forall k,\ Ax = b,\ x \in \{0,1\}^n. \tag{5}$$

Consequence: linear approximation results in

1. piecewise tangent approximation ($\Rightarrow$ outer bound for feasible solutions)
2. piecewise linear approximation ($\Rightarrow$ inner bound for feasible solutions)

Using the Taylor approximation of $H(z)^2$ at z_k and linearizing the quadratic terms, we provide its piecewise-tangent approximation,

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\begin{aligned}
\min \langle X, Q\rangle + c^T x \quad \text{s.t.} \quad & \langle \Sigma_k, Z^k\rangle \le d_k^2 - 2d_k\mu_k^T x + \langle \mu_k\mu_k^T, X\rangle \ \forall k \\
& \mu_k^T x \le d_k \ \forall k \\
& \hat{F}^k - (1 - X_{ij})U^+ \le Z_{ij}^k \le \hat{F}^k \ \forall i, j, k \\
& 0 \le Z_{ij}^k \le X_{ij}U^+ \ \forall i, j, k \\
& a_l + b_l z_k \le \hat{F}^k \ \forall k, l \\
& x_i + x_j - 1 \le X_{ij} \le \min\{x_i, x_j\} \ \forall i, j \\
& X_{ii} = x_i \ \forall i, \ \ X_{ij} \ge 0 \ \forall i, j \\
& \sum_{k=1}^{K} z_k = 1, \ \ z_k \ge 0 \ \forall k, \ \ Ax = b, \ \ x \in \{0,1\}^n.
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This model is a relaxation of problem (4) as shown by the following lemma:

Lemma 22

The optimal value ϕ_N^* of the previous model is a lower bound of (4). Moreover if the trivial solution $x = 0$ is not feasible to (4) and tangents points are uniformly selected on the interval $(0, 1]$, then $\lim_{N \to +\infty} \phi_N^* = \phi^*$ where ϕ^* is the optimal value of (4).

When we approximate $H(z)^2$ by using the piecewise-linear technique, then we have **another mixed integer linear program which is a restriction of the problem (4)**

Lemma 23

The optimal value ϕ_N^* of the restriction problem is an upper bound of (4). Moreover if the trivial solution $x = 0$ is not feasible to (4) and the interpolation points are uniformly selected on the interval $(0, 1]$, then $\lim_{N \to +\infty} \phi_N^* = \phi^*$ where ϕ^* is the optimal value of (4).

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$$\min x^T Qx + c^T x \quad \text{subject to} \quad \mathbb{P}\{Tx \le d\} \ge p, \; Ax = b, \; x \in \{0,1\}^n$$

SDP relaxation...

$$\min \langle X, Q \rangle + c^T x \quad \text{s.t.} \quad \begin{pmatrix} (d_k - \mu_k^T x) I & \Sigma_k^{1/2} \tilde{z}_k \\ \tilde{z}_k{}^T (\Sigma_k^{1/2})^T & d_k - \mu_k^T x \end{pmatrix} \succeq 0 \; \forall k$$

$$\tilde{z}_{ki} \ge a_l x_i + b_l z_{ki} \; \forall i, l$$

$$\sum_{k=1}^{K} z_{ki} = x_i, \; z_{ki} \ge 0 \; \forall i, k$$

$$\tilde{F}_k - (1 - x_i) M^+ \le \tilde{z}_{ki} \le M^+ x_i \; \forall i, k$$

$$0 \le \tilde{z}_{ki} \le \tilde{F}_k, \forall k, i, \; a_l + b_l z_k \le \tilde{F}_k \; \forall l, k$$

$$A_t^T x = b_t, \; A_t^T X A_t = b_t^2, \; \forall t$$

$$\sum_{k=1}^{K} z_k = 1, \; z_k \ge 0 \; \forall k$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \; X \ge 0.$$

1. the problems were solved by **Sedumi**
2. all tests ran on a Pentium(R)D @ 3.00 GHz with 2.0 GB RAM
3. 3 instances taken from the OR-library with 10 constrained resources

Instances and parameters

$[10, 20]$ ... matrix Q interval generation
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Table: Computational results when $p = 0.9$

$\theta = 1$	$V^{MILP}(C)$ e+03	CPU (s)	$V^{MILP}(R)$ e+03	CPU (s)	Gap (%)	V^{LP} e+03	CPU (s)	Gap (%)	V^{SDP} e+03	CPU (s)	GAP (%)
(14,5,5)	2.731	0.80	2.911	0.39	6.59	3.668	0.03	34.31	3.156	0.56	15.56
(28,10,5)	6.734	29.31	6.899	16.15	2.45	9.365	0.44	39.07	7.535	3.13	11.89
(50,5,10)	8.025	312.61	8.025	251.79	0.00	11.19	9.31	39.56	9.271	92.75	15.53

$\theta = 2$	$V^{MILP}(C)$ e+03	CPU (s)	$V^{MILP}(R)$ e+03	CPU (s)	Gap (%)	V^{LP} e+03	CPU (s)	Gap (%)	V^{SDP} e+03	CPU (s)	GAP (%)
(14,5,5)	2.731	0.76	2.911	0.36	6.59	3.668	0.03	34.31	3.178	0.60	16.37
(28,10,5)	7.361	15.73	7.361	16.60	0.00	9.369	0.45	27.28	7.725	3.64	4.94
(50,5,10)	8.025	253.88	8.025	228.29	0.00	11.20	7.35	39.56	9.513	92.32	18.54

$\theta = 5$	$V^{MILP}(C)$ e+03	CPU (s)	$V^{MILP}(R)$ e+03	CPU (s)	Gap (%)	V^{LP} e+03	CPU (s)	Gap (%)	V^{SDP} e+03	CPU (s)	GAP (%)
(14,5,5)	2.731	0.47	2.731	0.39	0.00	3.668	0.03	34.31	3.167	0.64	15.96
(28,10,5)	7.361	10.76	7.361	14.87	0.00	9.366	0.43	27.24	7.830	3.14	6.37
(50,5,10)	8.025	269.26	8.025	103.62	0.00	11.19	5.21	39.44	9.594	85.47	19.55

Conclusions

1. Handling joint chance constraints with continuous distributions with dependent random variables, quadratic objective function, and $0-1$ variables is a challenging problem.
2. Using the approaches presented above, we solved linear and quadratic problems with $0-1$ variables using conic optimization formulations.
3. Recently, we introduced chance constraints in stochastic games.
4. A high number of results are currently published on distributionally robust chance constrained problems.

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